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MIXING PROPERTIES OF MARKOV OPERATORS AND ERGODIC TRANSFORMATIONS, AND ERGODICITY OF CARTESIAN PRODUCTS

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Dedicated to the memory of Shlomo Horowitz

ABSTRACT

Let T be a Markov operator on $L_1(X, \Sigma, m)$ with $T^* = P$. We connect properties of P with properties of all products $P \times Q$, for Q in a certain class: (a) (Weak mixing theorem) P is ergodic and has no unimodular eigenvalues $\neq 1$ $\Leftrightarrow$ for every Q ergodic with finite invariant measure $P \times Q$ is ergodic $\Leftrightarrow$ for every $u \in L_1$ with $\int u\,dm = 0$ and every $f \in L_\infty$ we have $N^{-1}\sum_{n=1}^{N} |\langle u, P^n f\rangle| \to 0$. (b) For every $u \in L_1$ with $\int u\,dm = 0$ we have $\|T^n u\|_1 \to 0$ $\Leftrightarrow$ for every ergodic Q, $P \times Q$ is ergodic. (c) P has a finite invariant measure equivalent to m $\Leftrightarrow$ for every conservative Q, $P \times Q$ is conservative. The recent notion of mild mixing is also treated.

1. Introduction

In the ergodic theory of measure preserving transformations of a finite measure space there is a fairly well understood hierarchy of mixing conditions: ergodicity, weak mixing, mixing, mixing of all orders, K-automorphisms, B-shifts, and various other intermediate concepts. Various attempts have been made to extend some of these notions to transformations, and more generally to Markov operators, that preserve an infinite measure (cf. [13], [14], [16]–[20]). In particular the Koopmans–von Neumann–Halmos (K–vN–H) theorem, which says that (for probability preserving transformations) weak mixing is equivalent to a condition on the spectrum as well as to a multiplier property (T is weak mixing if and only if for all S ergodic $T \times S$ is ergodic), presents a challenge to find an appropriate analogue. It was this challenge that motivated much of the work that we shall now describe.

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Let (X, Σ, m) be a σ-finite measure space, and let P be a Markov operator on $L_\infty(m)$, i.e., P is the adjoint of a positive contraction T on $L_1(X, \Sigma, m)$ (see [7] for the properties and definitions that won't be made explicitly in what follows). Recall that P is said to be *ergodic* if for all $u \in L_1$ with zero integral ($\int u dm = 0$) we have

$$\lim_{N\to\infty} \left\| \frac{1}{N} \sum_{n=1}^{N} T^n u \right\|_1 = 0. \tag{1}$$

By the Hahn–Banach theorem P is ergodic if and only if $Pf = f$ implies that f is a constant. Next we shall say that P is *weakly mixing* if for all u with zero integral

$$\lim_{N\to\infty} \sup_{\|f\|_\infty \le 1} \frac{1}{N} \sum_{n=1}^{N} |\langle u, P^n f\rangle| = 0. \tag{2}$$

In [13] it is shown that this is equivalent to

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |\langle T^n u, f\rangle| = 0 \tag{3}$$

for all $f \in L_\infty$ and all u with zero integral. We say that P is *mixing* if for all u with zero integral $T^n u \to 0$ weakly in L_1, and that P is *completely mixing* if for all such u, $\|T^n u\|_1 \to 0$. In [16] it is shown that if P has no finite invariant measure and is mixing then P is completely mixing; thus an invertible ergodic transformation with infinite invariant measure cannot be mixing with this definition of mixing. We shall also investigate a mixing notion recently introduced in [8] for point transformations with finite invariant measure, namely: P is said to be *mildly mixing* if $P^{n_i} f \to f$ weak-* in L_∞ implies that f is constant a.e. Since P is mixing if and only if all weak-* limit points of $\{P^n f\}$ are constants [17], if P is mixing it is mildly mixing.

The cartesian product of two Markov operators P on $L_\infty(X, \Sigma, m)$ and Q on $(Y, \mathcal{F}, \mu)$ can be defined from $P(x, A)$ and $Q(y, B)$, the transition probabilities of P and Q, by using $P(x, \cdot) \times Q(y, \cdot)$ to define a transition probability on $(X \times Y, \Sigma \times \mathcal{F})$ and so $P \times Q$ is again a Markov operator. After these definitions we can describe the main results that will be presented here. It turns out that we do not need to assume existence of a σ-finite invariant measure.

Generalizing the K–vN–H weak mixing theorem we show in §4 that P is weakly mixing if and only if "for all ergodic Q with finite invariant measure $P \times Q$ is ergodic" if and only if "P is ergodic and has no unimodular eigenvalues other than 1." Complete mixing also is a multiplier property, namely: P is completely mixing if and only if for every ergodic Q, $P \times Q$ is ergodic (Theorem

5.1). A sample result relating mild mixing to a multiplier property is Corollary 6.4 which says that if P is mildly mixing and Q is ergodic and conservative with σ-finite invariant measure then $P \times Q$ is ergodic. Section 3 treats in a sense a more fundamental question and shows that for any conservative P there always are Q such that $P \times Q$ is conservative. Conversely, if P preserves no finite measure then there always is a conservative ergodic Q such that $P \times Q$ is not conservative. Finally, or firstly, we give a rapid discussion in §2 of the construction of a point transformation, the Markov shift, associated with a Markov operator and in particular compare the various mixing properties of the operator with those of the corresponding shift.

All multiplier-mixing questions have by no means been resolved. We should like to mention in particular the following: What is the mixing property (if any) of P that is equivalent to the "multiplier property": for any ergodic Q such that $P \times Q$ is conservative, $P \times Q$ is also ergodic?

2. Ergodic properties of the Markov shift

Let (X, Σ) be a measurable space, and $P(x, A)$ a transition probability on $X \times \Sigma$. Define $\Omega = \Pi_{i=0}^{\infty} X_i$, with $X_i = X$ for each i, and let $\mathcal{B}$ be the σ-algebra generated by the cylinders. For $x \in X$ and $A_0, A_1, \cdots, A_k \in \Sigma$ define

$$P_x(A_0 \times A_1 \times \cdots \times A_k) = I_{A_0} P(1_{A_1} P(\cdots I_{A_{k-1}} P(I_{A_k} 1) \cdots)(x)$$

(where I_A is the operator of multiplication by the indicator function 1_A, and $Pf(x) = \int f(y) P(x, dy)$). P_x is uniquely defined, since $P1 = 1$ and P is linear, and can be extended to a probability measure $P_x(\cdot)$ on $\mathcal{B}$. Denote $Q_B(x) = P_x(B)$ for $x \in X$, $B \in \mathcal{B}$. Then $\{B \in \mathcal{B} : Q_B(\cdot)$ is Σ-measurable$\}$ is a monotone class containing all finite unions of disjoint cylinders, hence equals $\mathcal{B}$.

For an element $\omega \in \Omega$ we denote its n-th coordinate by $x_n(\omega)$. Then $x_n(\cdot)$ is a measurable map, and for any $g(x)$ bounded Σ-measurable, $h_n(\omega) = g(x_n(\omega))$ is $\mathcal{B}$-measurable.

For $h(\omega)$ bounded $\mathcal{B}$-measurable, we define

$$\hat{h}(x) = \int h(\omega) dP_x \equiv \int h(\omega) P_x(d\omega).$$

For $B \in \mathcal{B}$, $\hat{1}_B(x) = P_x(B) = Q_B(x)$, and by approximation we have that $\hat{h}(x)$ is Σ-measurable.

Now, given $g \in B(X,\Sigma)$, let $h(\omega) = g(x_0(\omega))$. We show that $\hat{h} = g$: If $g = 1_A$ with $A \in \Sigma$, then

$$\hat{h}(x) = \int 1_A(x_0(\omega))P_x(d\omega) = P_x(\{x_0(\omega) \in A\}) = 1_A(x).$$

Approximation yields the result.

We now define the shift transformation θ in Ω by $\theta(x_0, x_1, \cdots) = (x_1, x_2, \cdots)$, which is clearly $\mathcal{B}$-measurable and let $(Th)(\omega) = h(\theta\omega)$ be defined on $\mathcal{B}$-measurable functions.

LEMMA 2.1. $\widehat{T^n h} = P^n \hat{h}$, *for* $h \in B(\Omega, \mathcal{B})$.

PROOF. Let $B = A_0 \times A_1 \times A_2 \times \cdots \times A_k$, with $A_j \in \Sigma$. Then

$$\widehat{T^n 1_B}(x) = \hat{1}_{\theta^{-n}B}(x) = P_x(\theta^{-n}B) = P_x(X \times X \cdots X \times A_0 \times \cdots A_k)$$

$$= P^n I_{A_0} P(\cdots P(I_{A_k} 1) \cdots)(x).$$

Hence $\widehat{T^n 1_B} = P^n Q_B = P^n \hat{1}_B$. Now $\{B \in \mathcal{B} : \widehat{T^n 1_B} = P^n \hat{1}_B\}$ is monotone and contains finite unions of disjoint cylinders, hence equals $\mathcal{B}$. Linearity and approximations finish the proof.

For a finite measure μ on Σ define $\tilde{\mu}$ on $\mathcal{B}$ by

$$\tilde{\mu}(B) = \int \hat{1}_B(x)\mu(dx) = \int P_x(B)\mu(dx).$$

Then $\tilde{\mu}$ is a measure, and $\langle \tilde{\mu}, h \rangle = \langle \mu, \hat{h} \rangle$ for $h \in B(\Omega, \mathcal{B})$. The following lemma is now easy.

LEMMA 2.2. (a) $\langle \tilde{\mu}, T^n h \rangle = \langle \mu, P^n \hat{h} \rangle$.

(b) *If* $\mu \ll m$, *then* $\tilde{\mu} \ll \tilde{m}$.

(c) *If* $mP \ll m$, *then* $\tilde{m}\theta^{-1} \ll \tilde{m}$.

For the rest of this section, we assume $mP \ll m$. Then P induces a Markov operator on $L_\infty(m)$ (still denoted by P), and θ is $\tilde{m}$-non-singular. We assume that $m(X) = 1$.

THEOREM 2.3. *Let* C *and* D *be the conservative and dissipative parts for* P. *The conservative and dissipative parts for* θ *are* $\{\omega : x_0(\omega) \in C\}$ *and* $\{\omega : x_0(\omega) \in D\}$, *respectively.*

PROOF. Let $\tilde{D}$ be the dissipative part for θ.

It is known [11] that $D = \bigcup_{k=1}^{\infty} A_k$ with $\sum_{n=0}^{\infty} P^n 1_{A_k} \in L_\infty(m)$. Let $B_k = \{\omega : x_0(\omega) \in A_k\}$. Then $\hat{1}_{B_k} = 1_{A_k}$, and

$$\left\langle \tilde{m}, \sum_{n=0}^{\infty} T^n 1_{B_k} \right\rangle = \sum_{n=0}^{\infty} \langle \tilde{m}, T^n 1_{B_k} \rangle = \sum_{n=0}^{\infty} \langle m, P^n \hat{1}_{B_k} \rangle$$

$$= \left\langle m, \sum_{n=0}^{\infty} P^n 1_{A_k} \right\rangle < \infty.$$

Hence $\sum_{n=0}^{\infty} T^n 1_{B_k} < \infty$ $\tilde{m}$-a.e., and $B_k \subset \tilde{D}$. Hence $\{\omega : x_0(\omega) \in D\} = \bigcup_{k=1}^{\infty} B_k \subset \tilde{D}$. Now let $B \subset \tilde{D} - \{x_0(\omega) \in D\}$, such that $\sum_{n=0}^{\infty} T^n 1_B \in L_\infty(\tilde{m})$. Hence

$$\left\langle m, \sum_{n=0}^{\infty} P^n \hat{1}_B \right\rangle = \sum_{n=0}^{\infty} \langle m, P^n \hat{1}_B \rangle = \sum_{n=0}^{\infty} \langle \tilde{m}, T^n 1_B \rangle < \infty.$$

Hence $\sum_{n=0}^{\infty} P^n \hat{1}_B < \infty$ a.e., and $\{x : \hat{1}_B(x) > 0\} \subset D$. But $B \subset \{x_0(\omega) \in C\}$, so $\hat{1}_B \leqq 1_C$, so it is zero on D. Hence $\tilde{m}(B) = \langle m, \hat{1}_B \rangle = 0$. Q.E.D.

For the next results, we need the following formula:

Lemma 2.4. *For $f \in B(\Omega, \mathcal{B})$ and $A_0, A_1, \cdots, A_k \in \Sigma$,*

$$\int 1_{A_0 \times A_1 \times \cdots \times A_k}(\omega) f(\theta^{k+1}\omega) \tilde{m}(d\omega) = \int I_{A_0} P(I_{A_1} P(\cdots I_{A_k} P\hat{f}) \cdots) dm.$$

Proof. Take first f an indicator function of a cylinder, and apply the definitions. Then use linearity and approximation.

Theorem 2.5. *P is ergodic $\Leftrightarrow$ θ is ergodic.*

Proof. (a) Let P be ergodic. Let $h \in B(\Omega, \mathcal{B})$ satisfy $Th = h$ $\tilde{m}$-a.e. For every finite $\mu \ll m$ we have

$$\langle \mu, \hat{h} \rangle = \langle \tilde{\mu}, h \rangle = \langle \tilde{\mu}, Th \rangle = \langle \mu, \widehat{Th} \rangle = \langle \mu, P\hat{h} \rangle,$$

so that $P\hat{h} = \hat{h}$ a.e., and $\hat{h} = \text{const.}$ a.e. Let $\hat{h} \equiv \alpha$.

Let ν be the measure on $\mathcal{B}$ defined by $d\nu = h d\tilde{m}$. Then, using Lemma 2.4,

$$\nu(A_0 \times A_1 \times \cdots \times A_k) = \int 1_{A_0 \times A_1 \times \cdots \times A_k}(\omega) h(\omega) \tilde{m}(d\omega)$$

$$= \int 1_{A_0 \times \cdots \times A_k}(\omega) h(\theta^{k+1}\omega) \tilde{m}(d\omega)$$

$$= \int (I_{A_0} P(\cdots I_{A_k} P\hat{h}) \cdots) dm$$

$$= \alpha \tilde{m}(A_0 \times A_1 \times \cdots \times A_k).$$

Hence $\nu = \alpha\tilde{m}$ and $h = \alpha\ \tilde{m}$-a.e.

(b) Assume now that θ is ergodic. If $\mu \ll m$ is a finite signed measure with $\mu(X) = 0$, then $\tilde{\mu} \ll \tilde{m}$ and $\tilde{\mu}(\Omega) = 0$. Hence by ergodicity

$$\left\| N^{-1} \sum_{n=1}^{N} \mu\theta^{-n} \right\| \xrightarrow[N\to\infty]{} 0.$$

For $A \in \Sigma$, let $\tilde{A} = \{\omega : x_0(\omega) \in A\}$. Then

$$N^{-1} \sum_{n=1}^{N} \langle \mu, P^n 1_A \rangle = N^{-1} \sum_{n=1}^{N} \tilde{\mu}\theta^{-n}(\tilde{A}) \xrightarrow[N\to\infty]{} 0,$$

since $\hat{1}_{\tilde{A}} = 1_A$. Hence P is ergodic.

THEOREM 2.6. *P on $L_\infty(m)$ has the same unimodular eigenvalues as T on $L_\infty(\tilde{m})$.*

PROOF. (a) Let $|\lambda| = 1$, $\lambda \neq 1$, be an eigenvalue of T. There exists $0 \not\equiv h \in L_\infty(\tilde{m})$ with $Th = \lambda h$ $\tilde{m}$-a.e. For $\mu \ll m$ we have

$$\lambda \langle \mu, \hat{h} \rangle = \lambda \langle \tilde{\mu}, h \rangle = \langle \tilde{\mu}, Th \rangle = \langle \mu, P\hat{h} \rangle.$$

Hence $P\hat{h} = \lambda\hat{h}$. We show that $\hat{h} \not\equiv 0$ (mod m). Let ν be the finite complex measure on $\mathcal{B}$ defined by $d\nu = hd\tilde{m}$. We obtain

$$\lambda^{k+1}\nu(A_0 \times A_1 \times \cdots \times A_k) = \int I_{A_0} P(I_{A_1} \cdots (I_{A_k} P\hat{h}) \cdots) dm.$$

Hence, if $\hat{h} \equiv 0$, $\nu = 0$, and $h = d\nu/d\tilde{m} = 0$ a.e., a contradiction. Thus, λ is an eigenvalue of P.

(b) Let $|\lambda| = 1$, $\lambda \neq 1$, be an eigenvalue of P. Hence there exists a finite complex measure $\mu \ll m$ such that $N^{-1}\sum_{n=1}^{N} \lambda^{-n}\mu P^n$ does not converge to zero (if $Pg = \lambda g$, $g \not\equiv 0$, take μ with $\int g d\mu \neq 0$).

Assume λ is not an eigenvalue of T. Then $\tilde{\mu}$ is orthogonal to the fixed points of $\lambda^{-1}T$ (there are none), so that $\|N^{-1}\sum_{n=1}^{N} \lambda^{-n}\tilde{\mu}\theta^{-n}\| \to 0$, and $N^{-1}\sum_{n=1}^{N} \lambda^{-n}\mu P^n \to 0$ weakly is shown as in the previous proof, a contradiction.

REMARK. It now follows immediately that the unimodular eigenvalues of P are a subgroup of the unit circle.

THEOREM 2.7. *P on $L_\infty(m)$ is weakly mixing $\Leftrightarrow$ T is weakly mixing on $L_\infty(\tilde{m})$.*

PROOF. (a) Let T be weakly mixing. Let $\mu \ll m$ be a finite signed measure with $\mu(X)=0$. Then $\tilde{\mu} \ll \tilde{m}$ and $\tilde{\mu}(\Omega)=0$. For $A \in \Sigma$, let $\tilde{A} = \{\omega : x_0(\omega) \in A\}$. Then, by Lemma 2.2,

$$N^{-1}\sum_{n=1}^{N} |\langle \mu, P^n 1_A \rangle| = N^{-1}\sum_{n=1}^{N} |\langle \tilde{\mu}, T^n 1_{\tilde{A}} \rangle|,$$

which converges to 0 since T is weakly mixing. Hence P is weakly mixing.

(b) Let P be weakly mixing. We assume w.l.g. that $m(X)=1$. Hence also $\tilde{m}(\Omega)=1$. Let ν be a measure on $(\Omega, \mathcal{B}, \tilde{m})$ with $d\nu/d\tilde{m} = 1_{A_0\times A_1\times\cdots\times A_k}$. Let $\alpha = \tilde{m}(A_0 \times A_1 \times \cdots \times A_k)$. For $h \in L_\infty(\tilde{m})$ we have

$$\begin{aligned}\langle \nu - \alpha\tilde{m}, T^{k+r}h\rangle &= \int 1_{A_0\times A_1\times\cdots\times A_k}(\omega) h(\theta^{k+r}\omega)\tilde{m}(d\omega) - \alpha \int h(\theta^{k+r}\omega)\tilde{m}(d\omega) \\ &= \int I_{A_0}(PI_{A_1}(\cdots I_{A_k}P^r\hat{h})\cdots)dm - \alpha\int P^{k+r}\hat{h}dm \\ &= \langle \hat{\nu} - \alpha m P^k, P^r\hat{h}\rangle,\end{aligned}$$

where $\hat{\nu} = (\cdots((mI_{A_0})PI_{A_1})\cdots)PI_{A_k}$. By the definitions, $\hat{\nu}(X)=\alpha$. Hence $(\hat{\nu} - \alpha m P^k)(X) = 0$, and by weak mixing of P we have

$$N^{-1}\sum_{r=1}^{N} |\langle \nu - \alpha\tilde{m}, T^{k+r}h\rangle| = N^{-1}\sum_{r=1}^{N} |\langle \hat{\nu} - \alpha m P^k, P^r\hat{h}\rangle| \xrightarrow[N\to\infty]{} 0.$$

Hence $N^{-1}\Sigma_{r=1}^{N}|\langle \nu - \alpha\tilde{m}, T^r h\rangle| \to 0$ for every $h \in L_\infty(m)$. Standard approximations yield that T is weakly mixing.

THEOREM 2.8. *P on $L_\infty(m)$ is mixing (completely mixing) $\Leftrightarrow$ T on $L_\infty(\tilde{m})$ is mixing (completely mixing).*

The proof is similar to the previous proof. The result for complete mixing is essentially due to Jamison and Orey [12]. The mixing case is well-known.

Furstenberg and Weiss [8] have introduced the concept of *mild mixing* for (invertible) ergodic transformations with finite invariant measure. We now have the following

THEOREM 2.9. *P on $L_\infty(m)$ is mildly mixing $\Rightarrow$ T on $L_\infty(\tilde{m})$ is mildly mixing. If $L_1(m)$ is separable, also the converse is true.*

PROOF. (a) Let P be mildly mixing. Let $h \in L_\infty(\tilde{m})$ satisfy $T^{n_i}h \to h$ weak-*, for some $\{n_i\}$. Then for any finite measure $\mu \ll m$, we have by Lemma 2.4 that

$$\langle \mu, P^{n_i}\hat{h}\rangle = \langle \tilde{\mu}, T^{n_i}h\rangle \to \langle \tilde{\mu}, h\rangle = \langle \mu, \hat{h}\rangle.$$

Hence $P^{n_i}\hat{h} \to \hat{h}$ weak-* in $L_\infty(m)$, so $\hat{h}$ is constant a.e. m. The first part of the proof of Theorem 2.5 shows that h is constant a.e. $\tilde{m}$. Hence T is mildly mixing.

(b) Let T be mildly mixing. If $g \in L_\infty(m)$ satisfies $P^{n_i}g \to g$ weak-* in $L_\infty(m)$, we look at $h \in L_\infty(\tilde{m})$ such that $\hat{h} = g$.

We now use the separability of $L_1(\tilde{m})$, implied by that of $L_1(m)$. Take a subsequence of $\{T^{n_i}h\}$ which converges weak-* in $L_\infty(m)$. By passing to the subsequence, we may and do assume that $T^{n_i}h$ converges weak-*, say to f. Hence, for each j,

$$T^{n_i+j}h \xrightarrow[i\to\infty]{} T^j f \qquad \text{(weak-*)}.$$

Fix k, and for $\nu \ll \tilde{m}$ with $d\nu/d\tilde{m} = 1_{A_0\times A_1\times\cdots\times A_k}$ we have, by Lemma 2.4, that for $j > k$,

$$\langle \nu, T^{n_i+j}h\rangle = \langle \hat{\nu}, P^{n_i+j-k-1}g\rangle \xrightarrow[i\to\infty]{} \langle \hat{\nu}, P^{j-k-1}g\rangle = \langle \nu, T^j h\rangle,$$

where $\hat{\nu} = mI_{A_0}PI_{A_1}\cdots I_{A_k}P$.

Now, for $j > k$, $\langle \nu, T^j f\rangle = \langle \nu, T^j h\rangle$, so also $\langle \nu, T^{n_i}f\rangle = \langle \nu, T^{n_i}h\rangle$ for all large i. Hence

$$\lim_{i\to\infty}\langle \nu, T^{n_i}f\rangle = \lim\langle \nu, T^{n_i}h\rangle = \langle \nu, f\rangle.$$

It now follows by linearity and approximation that $T^{n_i}f \to f$ weak-* in $L_\infty(\tilde{m})$, hence f is constant, say $f = \alpha$ $\tilde{m}$-a.e., by mild mixing of T. For $\mu \ll m$ a probability,

$$\langle \mu, g\rangle = \lim\langle \mu, P^{n_i}g\rangle = \lim\langle \tilde{\mu}, T^{n_i}h\rangle = \langle \tilde{\mu}, f\rangle = \alpha.$$

Hence $g = \alpha$ a.e., and P is mildly mixing.

We shall need in the sequel the following well-known lemma.

LEMMA 2.10. *Let σ be conservative (ergodic) on (X, m), τ non-singular on (Y, μ). If σ is mapped onto τ, i.e., there exists ρ measurable from X onto Y such that $\rho\sigma = \tau\rho$ and $m\rho^{-1} = \mu$, then τ is conservative. (ergodic).*

PROOF. If $f \in L_\infty(Y)$ satisfies $f(\tau y) \leqq f(y)$ a.e., define $g(x) = f(\rho x)$. Then $g(\sigma x) = f(\rho\sigma x) = f(\tau\rho x) \leqq f(\rho x) = g(x)$. Since σ is conservative, $g(\sigma x) = g(x)$ a.e. Hence $f(\tau y) = f(y)$ a.e. Hence σ is conservative. Ergodicity is proved similarly.

Now let μ be a σ-finite invariant measure for P (i.e., $\int Pf d\mu = \int f d\mu$ for $0 \leqq f \in B(X, \Sigma)$); if μ is finite, $\tilde{\mu}$ is a finite invariant measure for θ, by Lemma 2.2. If μ is σ-finite and infinite, $\tilde{\mu}$ can still be defined on $\mathcal{B}$, and will be σ-finite and invariant for θ. Let $\Omega_1 = \Pi_{i=-\infty}^{\infty} X_i$, with $X_i = X$ for every i, and let $\mathcal{B}_1$ be the σ-algebra generated by the cylinders. Let $A_0, A_1, \cdots, A_k \in \Sigma$. We look at the cylinder in $\mathcal{B}_1$,

$$B = \{x_j \in A_0,\ x_{j+1} \in A_1, \cdots, x_{j+k} \in A_k\},$$

and define $\tilde{\tilde{\mu}}(B) = \tilde{\mu}(A_0 \times A_1 \times \cdots \times A_k)$. The invariance of $\tilde{\mu}$ under θ makes $\tilde{\tilde{\mu}}$ well-defined, and it can be extended to a σ-finite measure on $\mathcal{B}_1$. Let σ be the *two-sided shift* $\sigma(x_i)_{i=-\infty}^{\infty} = (x_{i+1})_{i=-\infty}^{\infty}$. We obtain that $\tilde{\tilde{\mu}}\sigma^{-1} = \tilde{\tilde{\mu}}$, and have the following well-known result.

THEOREM 2.11 [10]. *Let P have a σ-finite invariant measure μ. P is conservative and ergodic $\Leftrightarrow$ the two sided shift σ is conservative and ergodic.*

REMARK. If P is ergodic and dissipative, σ will not be conservative; if it were, the shift θ would be conservative by Lemma 2.10 — contradicting Theorem 2.3. Since σ is invertible non-conservative (on a non-atomic space), it is not ergodic.

3. Conservative Cartesian products

Let P and Q be conservative Markov operators on $L_\infty(X, m)$ and $L_\infty(Y, \mu)$, respectively. We know that $P \times Q$ need not be conservative (e.g., P is the two-dimensional random walk, Q is the one-dimensional random walk).

In this section we will be concerned with finding Q, for a given P, such that $P \times Q$ will be conservative.

DEFINITION. A sequence $\{u_n\}_{n=0}^{\infty}$ is called a *recurrent renewal sequence* if there exists a recurrent Markov chain such that $u_n = p_{11}^{(n)}$ (so $u_0 = 1$, and $\sum_{n=0}^{\infty} u_n = \infty$).

For $n \geqq 1$, we define, in that chain, $f_n = \Pr\{\text{first return to 1 at time } n\}$ and $f_n^{(k)} = \Pr\{k\text{-th return to 1 at time } n\}$. We then have $f_n^{(k)} = \sum_{j=1}^{n-1} f_j f_{n-j}^{(k-1)}$, and $u_n = \sum_{k=1}^{n} f_n^{(k)}$. Also, by recurrence, $\sum_{n=1}^{\infty} f_n = 1$. On the other hand, given a sequence $a_n \geqq 0$ for $n \geqq 1$, such that $\sum_{n=1}^{\infty} a_n = 1$, we may define $a_n^{(k)} = \sum_{j=1}^{n-1} a_j a_{n-j}^{(k-1)}$ (for $k \leqq n$), and $u_n = \sum_{k=1}^{n} a_n^{(k)}$. We then define $p_{1j} = a_j$, $p_{i,i-1} = 1$ for $i \geqq 2$, $p_{ij} = 0$ for the other entries. Then $f_n = a_n$, so that $p_{11}^{(n)} = u_n$.

LEMMA 3.1. (Brunel [3]). *If $b_n \geqq 0$ and $\lim_{n\to\infty} b_n = 0$, there exists a recurrent renewal sequence $\{u_n\}$, $0 < u_{n+1} \leqq u_n$, $u_{n+1}/u_n \uparrow 1$, such that $\sum_{n=0}^{\infty} u_n b_n < \infty$.*

THEOREM 3.2. *Let P be a Markov operator on $L_\infty(m)$. Then P has a finite invariant measure equivalent to m if and only if for every conservative Markov operator Q the Cartesian product $P \times Q$ is conservative.*

PROOF. (1) Assume P has no finite invariant measure m. There is a set A such that

$$\left\| N^{-1} \sum_{n=1}^{N} P^n 1_A \right\| \xrightarrow[N\to\infty]{} 0$$

(see [7]), and $N^{-1}\sum_{n=1}^{N} mP^n(A) \to 0$. Let $b_n = n^{-1}\sum_{j=1}^{n} mP^j(A)$, and let $w_n = mP^n(A)$. Let $\{u_n\}$ be the recurrent renewal sequence given by the lemma, and let $v_n = u_{2n}$. Then, for some chain, $v_n = p_{11}^{(2n)}$, and, since the chain with transition probabilities $q_{ij} = p_{ij}^{(2)}$ is also conservative, $\{v_n\}$ is a recurrent renewal sequence. Now, since $u_{n+1} \leqq u_n$,

$$\sum_{n=1}^{\infty} w_n v_n = \sum_{n=1}^{\infty} w_n u_{2n} \leqq \sum_{n=1}^{\infty} w_n 2 \sum_{k=n+1}^{2n} \frac{u_k}{k} \leqq 2 \sum_{n=1}^{\infty} w_n \sum_{k=n}^{\infty} \frac{u_k}{k}$$

$$= 2 \sum_{k=1}^{\infty} u_k k^{-1} \sum_{n=1}^{k} w_n = 2 \sum_{k=1}^{\infty} u_k b_k < \infty.$$

Hence $\int \sum_{n=1}^{\infty} v_n P^n 1_A dm = \sum_{n=1}^{\infty} v_n w_n < \infty$, so that $\sum_{n=1}^{\infty} v_n P^n 1_A(x) < \infty$ a.e.

Let q_{ij} be a recurrent Markov chain on $N = \{1, 2, 3, \cdots\}$ with $q_{11}^{(n)} = v_n$. Then, in $X \times N$, we have

$$\sum_{n=0}^{\infty} (P \times Q)^n 1_{A\times\{1\}}(x, 1) = \sum_{n=0}^{\infty} q_{11}^{(n)} P^n 1_A(x) < \infty,$$

so that $\sum_{n=0}^{\infty} (P \times Q)^n 1_{A\times\{1\}} < \infty$ a.e. on $A \times \{1\}$. Hence $P \times Q$ is not conservative: $A \times \{1\}$ is in its dissipative part.

(2) We adapt Flytzanis' proof [6] of the corresponding result for point transformations.

We may assume that m is invariant for P, $m(X) = 1$. Let Q on (Y, Σ, μ) be conservative; we assume $\mu(Y) = 1$. Denote $R = P \times Q$, and for $f \in L_\infty(X \times Y)$ we have

$$(Rf)(x, y) = \int f(u, v) P(x, du) Q(y, dv).$$

Assume that $0 \leqq f \in L_\infty(X \times Y)$ with $Rf \leqq f$ a.e. Define $h(y) = \int f(x, y)m(dx)$. Then

$$Qh(y) = \int h(v)Q(y, dv) = \int\int f(x, v)m(dx)Q(y, dv)$$

$$= \int\int\int f(u, v)P(x, du)m(dx)Q(y, dv) = \int Rf(x, y)m(dx)$$

$$\leqq \int f(x, y)m(dx) = h(y).$$

Since Q is conservative, $Qh(y) = h(y)$ a.e., and $\int [f(x, y) - Rf(x, y)]m(dx) = 0$ for a.e. y, so that $\int\int (f - Rf)dmd\mu = 0$. Hence $Rf \leqq f \Rightarrow Rf = f$, and by [7] this is equivalent to R being conservative.

Remark. If P is given by a point transformation without finite invariant measures, Q conservative, such that $P \times Q$ is not conservative, can be chosen to be also given by a point transformation: Take the Markov shift of the chain (q_{ij}) constructed in the first part of the proof. This construction is taken from [1]. By taking a two sided shift, we can have Q given by an invertible transformation. Note that we always construct Q with a σ-finite invariant measure.

Theorem 3.3. *Let σ be an invertible measurable transformation in (X, m), with m σ-finite and invariant for σ. If σ is conservative and ergodic, there exists a conservative and ergodic measure-preserving transformation τ on a σ-finite measure space (Y, Σ, μ) such that $\sigma \times \tau$ is conservative.*

Proof. Let $\theta = \sigma^{-1}$. Fix A with $0 < m(A) < \infty$. For $x \in X$ and $0 < t < 1$ define $u(t, x) = \sum_{n=0}^{\infty} t^n 1_A(\theta^n x)$. For a.e. x, $u(t, x) \uparrow \infty$ as $t \uparrow 1^-$, since also θ is conservative and ergodic. By Egorov's theorem, there is a set B_0 of positive measure such that

$$\alpha(t) = \inf\{u(t, x) : x \in B_0\} \xrightarrow[t \to 1^-]{} \infty.$$

Since $\alpha(t)$ is increasing on $[0, 1)$ and unbounded, there is a $0 \leqq g \in L_1[0, 1]$ such that $\int_0^1 \alpha(t)g(t)dt = \infty$, and $\int_0^1 g(t)dt = 1$.

$\{x \in X\colon \int_0^1 u(t, x)g(t)dt = \infty\}$ is θ-invariant (since $u(t, x) \leqq 1 + u(t, \theta x)$), and contains B_0. By ergodicity of θ, we have $\int_0^1 u(t, x)g(t)dt = \infty$ a.e. on X.

Let $u_n = \int_0^1 t^n g(t)dt$. Then $u_n \downarrow 0$, and

$$\sum_{n=0}^{\infty} u_n \geqq \sum_{n=0}^{\infty} u_n 1_A(\theta^n x) = \int_0^1 u(t, x)g(t)dt = \infty.$$

Now by the Schwartz–Cauchy inequality

$$u_{n+1}=\int_0^1 t^{n/2}t^{(n+2)/2}g(t)dt \leqq \left(\int_0^1 t^n g(t)dt\right)^{1/2}\left(\int_0^1 t^{n+2}g(t)dt\right)^{1/2}=\sqrt{u_n u_{n+2}}.$$

Hence $\{u_{n+1}/u_n\}$ is increasing, and $u_{n+1}/u_n \uparrow 1$.

CLAIM. For every B with $0<m(B)<\infty$ we have $\sum_{n=0}^{\infty} u_n 1_B(\theta^n x)=\infty$ a.e.

PROOF OF CLAIM. Since we have

$$\sum_{n=0}^{N} u_n 1_B(\theta^n x)=\sum_{n=1}^{N}(u_{n-1}-u_n)\sum_{k=0}^{n-1}1_B(\theta^k x)+u_N\sum_{k=0}^{N}1_B(\theta^k x)-1_B(x),$$

and by Hopf's theorem (Chacon–Ornstein's theorem for θ)

$$\sum_{n=0}^{N}1_A(\theta^n x)\Big/\sum_{n=0}^{n}1_B(\theta^n x)\to m(A)/m(B) \qquad \text{a.e.,}$$

$$\sum_{n=0}^{\infty}u_n 1_B(\theta^n x)<\infty \quad\Leftrightarrow\quad \sum_{n=0}^{\infty}u_n 1_A(\theta^n x)<\infty.$$

By Kaluza's theorem [15], $\{u_n\}$ is a recurrent renewal sequence. Let (q_{ij}) be an ergodic and conservative Markov chain such that $u_n=q_{11}^{(n)}$, and let τ be the (one-sided) Markov shift, which has a σ-finite invariant measure μ (on the path space Y). Let $Sf(x)=f(\tau x)$ be the Markov operator induced by τ, and $\hat{S}$ the dual Markov operator.

By Orey's theorem [20] (q_{ij}) has the strong ratio limit property. Let $\{y_0=1\}\equiv \Omega_0\subset Y$ be the set of paths starting at 1. By example 3.2 in [19], for every $F\subset\Omega_0$ we have

$$\lim_{n\to\infty}\mu(\Omega_0\cap(\tau^{-n}F))/\mu((\tau^{-n}\Omega_0)\cap\Omega_0)=\mu(F)/\mu(\Omega_0).$$

Also $\mu(\tau^{-n}\Omega_0)=u_n$ by the construction of the shift.

Now $\theta\times\hat{S}$ is a contraction of $L_1(m\times\mu)$. For $B\subset X$ with $0<m(B)<\infty$, we let $F=\{(x,y)\in B\times\Omega_0\colon\ \sum_{n=0}^{\infty}\hat{S}^n 1_{\Omega_0}(y)1_B(\theta^n x)<\infty\}$, and $F_x=\{y\in\Omega_0\colon (x,y)\in F\}$. Let $F_{x,k}=\{y\in\Omega_0\colon\ \sum_{n=0}^{\infty}1_B(\theta^n x)\hat{S}^n 1_{\Omega_0}(y)\leqq k\}$. Then

$$\begin{aligned} k\mu(F_x) &\geqq \sum_{n=0}^{\infty}1_B(\theta^n x)\int_{F_{x,k}}\hat{S}^n 1_{\Omega_0}(y)d\mu(y)\\ &=\sum_{n=0}^{\infty}1_B(\theta^n x)\langle \hat{S}^n 1_{\Omega_0},1_{F_{k,x}}\rangle\\ &=\sum_{n=0}^{\infty}1_B(\theta^n x)\mu(\Omega_0\cap\tau^{-n}F_{k,x}).\end{aligned}$$

Since $\mu(\Omega_0 \cap \tau^{-n}F_{k,x})/\mu(\Omega_0 \cap \tau^{-n}\Omega_0) \to \mu(F_{k,x})/\mu(\Omega_0)$, we obtain, if $\mu(F_{k,x}) > 0$, that, for a.e. $x \in B$, $\Sigma u_n 1_B(\theta^n x) < \infty$, a contradiction. Hence $\mu(F_x) = 0$ for a.e. $x \in B$, so that $(m \times \mu)(F) = 0$. Hence $B \times \Omega_0$ is in the conservative part of the Markov operator $(\theta \times S)^* = \sigma \times \tau$. Hence $X \times \Omega_0$ is in the conservative part of $\sigma \times \tau$, and so is, similarly, $X \times \tau^{-n}\Omega_0$. Hence $\sigma \times \tau$ is conservative.

THEOREM 3.4. *Let P be a conservative and ergodic Markov operator on $L_\infty(m)$, with m σ-finite invariant. Then there exists a conservative and ergodic Markov operator Q on l_∞ such that $P \times Q$ is conservative.*

PROOF. Let σ be the *two-sided* Markov shift of P, and let Q be the Markov chain constructed in Theorem 3.3, with shift τ such that $\sigma \times \tau$ is conservative. Let θ be the one-sided shift of P. Then $\sigma \times \tau$ is mapped onto $\theta \times \tau$ (with the respective measures), and by Lemma 2.10, $\theta \times \tau$ is conservative. $P \times Q$ is now conservative, since its shift is (isomorphic to) $\theta \times \tau$.

4. Ergodicity of Cartesian products and weak mixing

Let P be a Markov operator on $L_\infty(m)$, with m an invariant probability for P. It is well-known [9] that in this case, the following conditions are equivalent (T is the contraction on $L_1(m)$ with $T^* = P$):

(i) For every $u \in L_1(m)$ with $\int u\,dm = 0$ there exists a sequence $\{n_k\}$ such that $T^{n_k}u \to 0$ weakly in $L_1(m)$.

(ii) $P \times P$ is ergodic.

(iii) For every ergodic Markov operator Q with a finite invariant measure, $P \times Q$ is ergodic.

(iv) P is weakly mixing.

(v) P is ergodic, and has no unimodular eigenvalues $\neq 1$.

The existence of a finite invariant measure for P implies that P is conservative [7]. Each of the five conditions above implies that P is ergodic.

We would like to investigate the relationships among the above conditions, assuming only that P is ergodic. The trivial implications are (ii) $\Rightarrow$ (iv) $\Rightarrow$ (v). The implication (i) $\Rightarrow$ (iv) follows from a general Banach space result of Jones and Lin [13]. (iii) $\Rightarrow$ (v) is also easy.

We start by showing that (ii) does not imply (i), and (iv) does not imply (ii), even if P has a σ-finite invariant measure conservative. We then show that (iii), (iv) and (v) are equivalent. In short,

$$\text{(i)} \not\Leftarrow \text{(ii)} \Rightarrow \text{(iii)} \Leftrightarrow \text{(iv)} \Leftrightarrow \text{(v)} \Leftarrow \text{(i)}.$$

EXAMPLE 4.1. A conservative and ergodic contraction T on $L_1(m)$ with σ-finite invariant measure, such that $T \times T$ is conservative and ergodic (hence T is weak mixing), but there exists a function $v \in L_1(m)$, with $\int v dm = 0$, so that $\{T^n v\}$ has no subsequence converging weakly to zero.

CONSTRUCTION. Let θ be the one-sided shift of an aperiodic recurrent random walk on the integers, such that also $\theta \times \theta$ is conservative and ergodic (e.g., $P_{i,j} = \frac{1}{4}$ if $j = i \pm 1$, $P_{i,i} = \frac{1}{2}$).

Let $Tf(x) = f(\theta x)$. Then T is a contraction of $L_1(m)$ and $L_\infty(m)$, since the random walk has an invariant measure, and m is σ-finite non-finite. Also θ is exact, i.e, $\Sigma_\infty \equiv \bigcap_{n=1}^\infty \theta^{-n}\Sigma = \{\theta, \Omega\}$ (mod m), since the Markov operator of the transition probabilities is mixing.

Let $T^* = S$. S is also a contraction of L_1 and L_∞. If $f \in L_2$ is such that there are $f_n \in L_2$, $\|f_n\|_2 \leqq 1$, and $T^n f_n = f$, then f is Σ_∞-measurable, hence $f = 0$. Thus [17], $S^n \to 0$ strongly in L_2, hence for every A, B with $m(B) + m(A) < \infty$, $\langle T^n 1_A, 1_B \rangle = \langle S^N 1_B, 1_A \rangle \to 0$.

Next, note that $S \times S = (T \times T)^*$, so $T \times T$ and $S \times S$ are both conservative and ergodic [7].

Recall that a transformation θ preserving a σ-finite infinite measure m is called of *zero type* if $\int f(\theta^n x) g(x) dm \to 0$ for $f, g \in L_2(m)$. Thus, we have constructed a zero type transformation, and the next lemma finishes the example.

LEMMA 4.2. *If θ is of zero type, there exists a $v \in L_1(m)$ with $\int v dm = 0$, such that no subsequence of $\{v \circ \theta^n\}$ converges weakly in L_1 to zero.*

PROOF. Take A and B with $A \cap B = \varnothing$, $m(A) = m(B) = 1$, and define $v = 1_A - 1_B$. Denote $Tf(x) = f(\theta x)$.

Let N_r be such that for $n > N_r$,

$$m(\theta^{-n} B \cap A) = \langle T^n 1_B, 1_A \rangle < \frac{1}{2^{r+1}},$$

$$m(\theta^{-n} A \cap B) = \langle T^n 1_A, 1_B \rangle < \frac{1}{2^{r+1}}.$$

Let $\{n_k\}$ be an increasing subsequence. We take a further subsequence of it, so we may assume that $n_{k+1} - n_k > N_k$.

Let $E = \bigcup_{j=1}^\infty \theta^{-n_j} A$. Then

$$\begin{aligned}\langle T^{n_j}v, 1_E\rangle &= \langle T^{n_j}1_A, 1_E\rangle - \langle T^{n_j}1_B, 1_E\rangle \\ &= m(\theta^{-n_j}A \cap E) - m(\theta^{-n_j}B \cap E) \\ &= m(\theta^{-n_j}A) - m(\theta^{-n_j}B \cap E) \\ &= 1 - m(\theta^{-n_j}B \cap E).\end{aligned}$$

We conclude by showing $m(\theta^{-n_j}B \cap E) \leqq \frac{1}{2}$.

$$\begin{aligned}m(\theta^{-n_j}B \cap E) &= m\left(\theta^{-n_j}B \cap \left(\bigcup_{k=1}^{\infty} \theta^{-n_k}A\right)\right) \\ &\leqq \sum_{k=1}^{\infty} m(\theta^{-n_j}B \cap \theta^{-n_k}A) \\ &= \sum_{k=1}^{j-1} m(\theta^{-n_j}B \cap \theta^{-n_k}A) + m(A \cap B) \\ &\quad + \sum_{k=j+1}^{\infty} m(\theta^{-n_j}B \cap \theta^{-n_k}A) \\ &= \sum_{k=1}^{j-1} \langle T^{n_j-n_k}1_B, 1_A\rangle + \sum_{k=j+1}^{\infty} \langle 1_B, T^{n_k-n_j}1_A\rangle \\ &\leqq \sum_{k=1}^{j-1} \frac{1}{2^{k+1}} + \sum_{k=j+1}^{\infty} \frac{1}{2^k} \\ &= \frac{1}{2},\end{aligned}$$

since for $j > k$, $n_j - n_k \geqq n_{k+1} - n_k$, and for $k > j$, $n_k - n_j \geqq n_k - n_{k-1}$.

Proposition 4.3. *Let θ be a non-singular transformation in (X, Σ, m). If $\theta \times \theta \times \theta$ is ergodic, then $\sigma = \theta \times \theta$ is weakly mixing.*

Proof. We may and do assume $m(X) = 1$. Let $u(x, y) \in L_1(m \times m)$, with $\iint u(x, y)dm(y)dm(x) = 0$.

Step 1. Assume $u \in L_\infty(m \times m)$, and $\int u(x, y)m(dx) = 0$ for a.e. y. Then for any $f \in L_\infty(m \times m)$, we have, using Schwartz–Cauchy inequality in $L_2(m(dy))$

$$\left|\int\int u(x, y)f(\theta^n x, \theta^n y)m(dx)m(dy)\right|^2$$

$$\leqq \left\{\int \left|\int u(x, y)f(\theta^n x, \theta^n y)m(dx)\right| m(dy)\right\}^2$$

REMARK. The construction can be adapted to obtain any ergodic index k, and is simpler than the one given in Kakutani and Parry [14].

THEOREM 4.4. *Let P be ergodic Markov operator on $L_\infty(m)$. Then* (iii) $\Rightarrow$ (iv).

PROOF. Let B be the unit ball of $L_\infty(m)$, with the w^* topology, and denote by σ the continuous map of B into itself defined by restricting P to B. We may and do assume $m(X)=1$, and let T be the contraction on $L_1(m)$ with $T^*=P$.

Fix $u \in L_1(m)$ with $\int u dm = 0$. We have to show that $N^{-1}\sum_{n=1}^{N}|\langle u, P^n h\rangle| \to 0$ for every $h \in L_\infty$, or, equivalently, for every $h \in B$.

Let μ be an *ergodic* invariant measure for σ. We shall show that $\int|\langle u,h\rangle|\,d\mu(h)=0$. Let R be defined on $L_1(B,\mu)$ by $Rg(f)=g(\sigma f)=g(Pf)$. Then R is a contraction of $L_1(B,\mu)$. Then $S=T\times R$ is an ergodic contraction of $L_1(X\times B, m\times\mu)$, by (iii). Define $w\in L_1(X\times B)$ by $w(x,h)=u(x)\langle u,h\rangle$. Then $\int\int \omega(x,h)dm(x)d\mu(h)=0$, and ergodicity of S yields

$$0=\lim_{N\to\infty}\left\| N^{-1}\sum_{n=1}^{N} S^n w\right\|_1=\lim_{N\to\infty}\int_B\int_X\left| N^{-1}\sum_{n=1}^{N} T^n u(x)R^n u(h)\right| dm(x)d\mu(h).$$

Let

$$F_N(h)=\int_X\left|N^{-1}\sum_{n=1}^{N} T^n u(x)R^n u(h)\right| m(dx)$$
$$=\int_X\left|\cdot N^{-1}\sum_{n=1}^{N} T^n u(x)\langle u,P^n h\rangle\right| dm.$$

Then $F_N(h)\geqq 0$, and we have obtained that $\|F_N\|_1\to 0$ in $L_1(\mu)$. Hence $F_N(h)\to 0$ in μ-measure, and there is a subsequence $\{N_j\}$ such that $F_{N_j}(h)\to 0$ a.e. μ. Fix $h\in B$ for which $F_{N_j}(h)\to 0$, and define $v_N(x)=N^{-1}\sum_{n=1}^{N}\langle u,P^n h\rangle T^n u(x)$.. Then $\|v_{N_j}\|_1\to 0$ in $L_1(m)$, and

$$N_j^{-1}\sum_{n=1}^{N_j}|\langle u,P^n h\rangle|^2=N_j^{-1}\sum_{n=1}^{N_j}\langle u,P^n h\rangle\langle T^n u,h\rangle=\int v_{N_j}(x)h(x)dm \xrightarrow[j\to\infty]{} 0.$$

Thus, $N_j^{-1}\sum_{n=1}^{N_j}|\langle u,P^n h\rangle|^2\to_{j\to\infty}0$ for μ a.e. $h\in B$. By invariance of μ, we have

$$\int|\langle u,h\rangle|^2 d\mu(h)=\int N_j^{-1}\sum_{n=1}^{N_j}|\langle u,P^n h\rangle|^2 d\mu\to 0.$$

Hence $|\langle u,h\rangle|=0$ a.e. Since u is continuous on B, $\{h\in B:\langle u,h\rangle\}=0$ contains the support of μ.

$$\leqq \int \left| \int u(x,y) f(\theta^n x, \theta^n y) m(dx) \right|^2 m(dy)$$

$$= \int \left\{ \int u(x,y) f(\theta^n x, \theta^n y) m(dx) \right\} \left\{ \int u(z,y) f(\theta^n z, \theta^n y) m(dz) \right\} m(dy)$$

$$= \int \int \int u(x,y) u(z,y) f(\theta^n x, \theta^n y) f(\theta^n z, \theta^n y) m(dx) m(dz) m(dy).$$

Now let $v(x,y,z) = u(x,y)u(z,y)$, $g(x,y,z) = f(x,y)f(z,y)$. Since

$$\int \int \int v(x,y,z) d(m \times m \times m) = \int \left| \int u(x,y) m(dx) \right|^2 m(dy) = 0,$$

we have, by ergodicity of $\theta_3 = \theta \times \theta \times \theta$, that $N^{-1} \Sigma_{n=1}^{N} \langle v, g(\theta_3^n(x,y,z)) \rangle \to 0$. The above computation yields $N^{-1} \Sigma_{n=1}^{N} |\langle u, f(\sigma^n(x,y)) \rangle|^2 \to 0$, hence also

$$N^{-1} \sum_{n=1}^{N} |\langle u, f(\sigma^n(x,y)) \rangle| \to 0, \qquad \text{for } f \in L_\infty(m \times m).$$

Step 2. Assume only $u(x,y) \in L_\infty(m \times m)$. Define $u_1(x,y) = u_1(y) = \int u(x,y) m(dx)$. Then $\int u_1(y) m(dy) = 0$. Let $u_2(x,y) = u(x,y) - u_1(y)$. Then $\int u_2(x,y) m(dx) = 0$ for almost every y. Clearly $u_1, u_2 \in L_\infty(m \times m)$.

Now $u = u_1 + u_2$, and for $f \in L_\infty(m \times m)$ we have

$$N^{-1} \sum_{n=1}^{N} |\langle u, f \circ \sigma^n \rangle| \leqq N^{-1} \sum_{n=1}^{N} |\langle u_1, f \circ \sigma^n \rangle| + N^{-1} \sum_{n=1}^{N} |\langle u_2, f \circ \sigma^n \rangle|.$$

Last term tends to 0 by step 1 applied to u_2. First one tends to 0 by changing roles of x and y in step 1, and applying it to u_1.

Step 3. If $u \in L_1(m \times m)$ with $\int\int u d(m \times m) = 0$, we can approximate u (in L_1) by $u_1 \in L_\infty(m \times m)$ with $\int\int u_1 d(m \times m) = 0$. Hence the proposition is proved.

To obtain an example such that (iv) does not imply (ii), we show how to construct θ such that $\theta \times \theta \times \theta$ is ergodic, $\theta \times \theta \times \theta \times \theta$ is not ergodic. The transformation $\sigma = \theta \times \theta$ will be the required example.

Let $u_n = (n+1)^{-1/3}$. Then $u_n \downarrow 0$ and $u_{n+1}/u_n \uparrow 1$. By Kaluza's theorem [15], $\{u_n\}_{n=0}^{\infty}$ is a recurrent renewal sequence, and the corresponding recurrent Markov chain $P = (p_{ij})$ is aperiodic (see §3), since $P_{11}^{(n)} = u_n > 0$ for every n. Let θ be the two-sided shift of P. Then θ is conservative and ergodic. Now $Q = P \times P \times P$ is recurrent, since $\Sigma q_{11}^{(n)} = \Sigma (p_{11}^{(n)})^3 = \Sigma (n+1)^{-1} = \infty$, but $P \times P \times P \times P$ is not recurrent (hence the invariant measure is infinite). Now $\theta \times \theta \times \theta$ is (isomorphic to) the two-sided shift of $P \times P \times P$ and is ergodic, $\theta \times \theta \times \theta \times \theta$ is the two-sided shift of a non-recurrent chain, so is not conservative, hence cannot be ergodic.

Thus, $\int |u| d\mu = 0$ for every ergodic invariant probability μ. But the extreme points of the set of invariant probabilities for σ are the ergodic invariant probabilities, and $|u| \in C(B)$, so $\{\nu \in C(B)^*: \nu \geqq 0, \nu(1) = 1, \int |u| d\nu = 0\}$ contains all σ-invariant probabilities, by the Krein–Milman theorem. R is also a positive contraction of $C(B)$, hence $\| N^{-1} \Sigma_{n=1}^{N} R^n |u| \|_\infty \to 0$, or

$$\sup_{h \in B} N^{-1} \sum_{n=1}^{N} |\langle u, P^n h \rangle| \xrightarrow[N \to \infty]{} 0,$$

which is weak mixing.

Flytzanis' main result [6] is wrong, as is shown by the result of [8], so it cannot be used to show that (v) $\Rightarrow$ (iii) (for conservative Markov operators). We now turn to proving (v) $\Rightarrow$ (iii).

A seemingly weaker condition than (iii) is: (iii)′ For every ergodic Markov operator Q on a *separable* space with finite invariant measure, $P \times Q$ is ergodic.

LEMMA 4.5. *Condition* (iii)′ *is equivalent to condition* (iii).

PROOF. Let P on $L_\infty(X, \Sigma, m)$ satisfy (iii)′. Assume $m(X) = 1$. Let Q be an ergodic Markov operator on $L_\infty(Y, \mu)$, with μ an invariant probability for Q.

Let $v \in L_1(Y, \mathcal{B}, \mu)$. Let $\mathcal{B}_0$ be the smallest sub-σ-algebra with respect to which v is measurable. $\mathcal{B}_0$ is countably generated, and we can find a countably generated σ-algebra $\mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{B}$ such that $L_\infty(Y, \mathcal{B}_1, \mu)$ is invariant under Q (see Doob's book [4, p. 209]). Let Q_1 be the Markov operator on $(Y, \mathcal{B}_1)$, and $\mu_1 = \mu \mid \mathcal{B}_1$. Clearly Q_1 is ergodic, with μ_1 invariant. Now $P \times Q_1$ is ergodic by (iii)′. Let T be the operator on $L_1(x)$ with $T^* = P$, R the operator on $L_1(Y, \mu)$ with $R^* = Q$. Let R_1 be on $L_1(Y, \mathcal{B}_1, \mu_1)$ with $R_1^* = Q_1$. Then $R_1 v = Rv$. If $u \in L_1(X)$ with $\int\int u(x)v(y) dm d\mu = 0$, then by ergodicity of $P \times Q_1$ we have

$$\left\| N^{-1} \sum_{n=1}^{N} T^n u R^n v \right\|_1 = \left\| N^{-1} \sum_{n=1}^{N} T^n u R_1^n v \right\|_1 \to 0.$$

Now let $f \in L_1(m \times \mu)$ with $\int\int f(x, y) dm d\mu = 0$.

For $\varepsilon > 0$, let $u_i \in L_1(X)$, $v_i \in L_1(Y)$ such that $\| \Sigma_{i=1}^{j} u_i v_i - f \|_1 < \varepsilon$. Then

$$f(x, y) = f - \sum u_i v_i + \sum \left(u_i(x) - \int u_i dm \right) v_i(y)$$

$$+ \sum \left(\int u_i dm \right) \left(v_i(y) - \int v_i d\mu \right) + \sum \left(\int u_i dm \right) \left(\int v_i d\mu \right).$$

The last sum is a constant function with integral close to $\int\int f(x,y)dmd\mu$. Hence

$$\limsup_{N\to\infty}\left\| N^{-1}\sum_{n=1}^{N}(T\otimes R)^n f\right\|_1 \leqq 2\varepsilon.$$

This shows that $P\times Q$ is ergodic.

PROPOSITION 4.6. *Let θ be a non-singular measurable transformation on a finite measure space (X,Σ,m). Let U be a unitary operator in a separable Hilbert space H, and let $F(x)$ be a measurable function from X into H, satisfying $F(\theta x)=UF(x)$ a.e. If θ has no unimodular eigenvalues $\neq 1$, then $UF(x)=F(x)=F(\theta x)$ a.e.*

PROOF. Note first that if $A\subset X$ is invariant for θ ($\theta A\subset A$ and $\theta(X-A)\subset X-A$) the restriction of θ to A satisfies all the hypotheses. Secondly, if $0\not\equiv|f|$ is finite and $f(\theta x)=\lambda f(x)$, there is also a solution in L_∞, since $A=\{x:|f(x)|\leqq k\}$ is invariant. Thus we assume no finite measurable solutions to $f(\theta x)=\lambda f(x)$ for $\lambda\neq 1$, $|\lambda|=1$.

By separability of H, $\|F(x)\|$ is measurable finite valued, and $UF(x)=F(\theta x)$ shows that it is invariant for θ. Thus we may restrict ourselves to invariant sets on which $\|F(x)\|\leqq k$. Thus we assume $\|F(x)\|\leqq k$.

Let $H_0=\{h\in H:\ \|N^{-1}\Sigma_{n=1}^{N}U^n h\|\to 0\}$. To prove the result, we show that $F(x)\perp H_0$ a.e.

Fix $h\in H_0$, and let $H_1=\mathrm{clm}\{U^n h:-\infty<n<\infty\}$. Let P be the orthogonal projection onto H_1, and define $F_1(x)=PF(x)$. Then $F_1(x)$ is measurable from X into H_1, and $UP=PU$ implies $F_1(\theta x)=UF_1(x)$ a.e.

By [5, part II, X.5.2], H_1 is isometrically isomorphic to $L_2(\Gamma,\eta)$, where $\Gamma=\{\lambda:|\lambda|=1\}$, and η is a positive finite Borel measure. U then corresponds to multiplication by the function λ. Hence we may and do assume that F_1 maps X into $L_2(\Gamma,\eta)$. By [5, part I, III.11.17], since $F_1(x)$ is m-integrable ($\|F_1(x)\|$ is bounded), there is a bi-measurable function $f(x,\lambda)$ such that $F_1(x)=f(x,\cdot)$ for a.e. x. Hence for x in a set of full measure, $f(\theta x,\lambda)=\lambda f(x,\lambda)$ for a.e. λ. Thus we have $\lambda f(x,\lambda)=f(\theta x,\lambda)$ for $m\times\eta$ a.e. (x,λ). This shows that for λ in a set of full η measure, $f(\theta x,\lambda)=\lambda f(x,\lambda)$ for a.e. x. Let $f_\lambda(x)=f(x,\lambda)$. Then $f_\lambda(\theta x)=\lambda f(x)$, and f_λ is finite a.e. Hence $f_\lambda=0$ m-a.e. for $\lambda\neq 1$. Since $\eta\{1\}=0$ ($h\in H_0$), we have a.e. $f(x,\lambda)=0$. Hence $F_1(x)=0$ a.e., or $F(x)\perp H_1$ a.e. Taking h_n dense in H_0, we obtain $F(x)\perp H_0$, showing $UF(x)=F(x)$ a.e.

REMARK. Considering the same "eigenoperator equation," A. Beck [2] showed that if θ is conservative, for a.e. x there is an $\{n_i\}$ such that $\|U^{n_i}F(x)-$

$F(x)\| \to 0$. If θ is not conservative, Beck's result fails: X is the set of integers, $\theta(j) = j + 1$, $H = l_2$, U the shift and $F(j) = e_j$.

THEOREM 4.7 (Weak mixing theorem). *Let P be an ergodic Markov operator. Then conditions* (iii), (iv) *and* (v) *are equivalent.*

PROOF. We have to prove only (v) $\Rightarrow$ (iii). Let Q be an ergodic Markov operator with finite invariant measure, let θ be the one-sided shift of P, and let σ_0 be the one-sided shift of Q, σ its two-sided shift. $P \times Q$ is ergodic if and only if $\theta \times \sigma_0$ is ergodic, and it is enough to prove that $\theta \times \sigma$ is ergodic (σ is also conservative and ergodic). By Theorem 2.6 also θ satisfies (v), and is ergodic by Theorem 2.5.

Thus, the problem is reduced to point-transformations, and σ invertible on Y, preserving a probability measure μ. Lemma 4.5 shows that we have to prove ergodicity only for separable $L_2(Y, \mu)$.

Let $f(\theta x, \sigma y) = f(x, y)$ a.e., with $|f(x, y)| \leqq 1$. Define F from X into $L_2(Y, \mu)$ by $F(x)(y) = f(x, y)$. Let U be the unitary operator in L_2 induced by (the invertible) σ^{-1}. Then $F(\theta x) = UF(x)$ for a.e. x, so by Proposition 4.6 $F(x) = UF(x)$ for a.e. x. Hence, for a.e. x, $f(x, \cdot)$ is invariant for σ, so by ergodicity of σ, it is constant a.e. Thus $f(x, y)$ does not depend on y, or $f(x, y) = f(x)$. Now $F(\theta x) = f(x)$, so f is constant by ergodicity of θ.

REMARK. Proposition 4.6 was also proved independently by Michael Keane.

COROLLARY 4.8. *If P is weakly mixing, and Q is weakly mixing mixing with finite invariant measure, then $P \times Q$ is weakly mixing.*

PROOF. Use condition (iii).

COROLLARY 4.9. *Let P be a conservative and ergodic Markov operator with σ-finite invariant measure.*

(a) *P is weakly mixing if and only if its dual Markov operator is weakly mixing.*

(b) *P is weakly mixing if and only if its two-sided shift is weakly mixing.*

PROOF. (a) Let P be weakly mixing, and let $\hat{P}$ be the dual Markov operator. If Q is ergodic with an invariant probability, $P \times \hat{Q}$ is conservative and ergodic, and so is $\hat{P} \times Q = (P \times \hat{Q})^\wedge$.

(b) is also proved using condition (iii) (and [10]).

REMARK. Even in the absence of a finite invariant measure, weak mixing is weaker than mild mixing. For example, let τ be mildly mixing, ρ invertible weak mixing with invariant probability which is not mild mixing. Then $\theta = \tau \times \rho$ is weak mixing, not mild mixing, and has no finite invariant measure if τ has none.

5. Ergodicity of Cartesian products and mixing

THEOREM 5.1. *P is completely mixing if and only if $P \times Q$ is ergodic for every ergodic Q.*

PROOF. (i) Let P be completely mixing in the (probability) space (X, Σ, m).

Take Q ergodic in $(Y, \mathcal{F}, \mu)$, and let T be the linear contraction in $L_1(m)$ with $T^* = P$. (Remember that we assume that P and Q are given by transition probabilities.)

Let $f(x, y) \in L_\infty(X \times Y)$ be invariant for $P \times Q$. Take $u \in L_1(m)$ with $\int u dm = 0$. Then, for a.e. y,

$$\left| \int_X u(x) f(x,y) dm(x) \right| = \left| \int_X u(x)[(P \times Q)^n f](x,y) dm(x) \right|$$
$$= \left| \int T^n u(x) f_{n,y}(x) dm(x) \right|$$
$$\leqq \| T^n u \|_1 \| f_{n,y} \|_\infty$$
$$\to 0,$$

since for $f_{n,y}(x) = \int f(x,t) Q^{(n)}(y, dt)$ we have $\| f_{n,y} \|_\infty \leqq \| f \|_\infty$.

Hence $\int u(x) f(x,y) dm(x) = 0$ for $u \in L_1(m)$ with $\int u dm = 0$, so that for any $u \in L_1(m)$ we have $\int u(x) f(x,y) dm(x) = (\int u dm) \int f(x,y) dm(x)$. Let $h(y) = \int_X f(x,y) dm(x)$, and take $v(x,y) \in L_1(X \times Y)$. Then, using Fubini's theorem,

$$\int_{X \times Y} v(x,y) f(x,y) d(m \times \mu)$$
$$= \int_Y \left[\int_X v(x,y) f(x,y) dm(x) \right] d\mu(y)$$
$$= \int_Y \left[\int_X v(x,y) dm(x) \right] \left[\int_X f(x,y) dm(x) \right] d\mu(y)$$
$$= \int_Y h(y) \int_X v(x,y) dm(x) d\mu(y)$$
$$= \int_{X \times Y} v(x,y) h(y) d(m \times \mu).$$

This shows that $f(x, y) = h(y)$, and $Qh = (P \times Q)f = f = h$. Hence $f(x, y)$ is constant, by ergodicity of Q, and $P \times Q$ is ergodic.

(ii) Let P satisfy the condition. To show that P is mixing, we have to show that if there is a sequence $\{f_n\}$ in L_∞ with $\|f_n\| \leqq 1$ such that $Pf_{n+1} = f_n$, then $f_n \equiv$ constant for each n [17].

Let $\{f_n\}$ be such a sequence. We take for Q the shift on the integers Z; by our assumption $P \times Q$ is ergodic on $X \times Z$. Define F on $X \times Z$ by $F(x, n) = f_n(x)$. Then $\|F\|_\infty \leqq 1$.

$$(P \times Q)F(x, n) = \int_Y \int_X F(t, k)P(x, dt)Q(n, dk) = \int_X F(t, n+1)P(x, dt)$$

$$= \int_X f_{n+1}(t)P(x, dt) = Pf_{n+1}(x) = f_n(x) = F(x, n).$$

Hence $F(x, n)$ is constant a.e., and for each n fixed, $f_n(x)$ is constant a.e. Hence P is completely mixing.

REMARK. A non-singular transformation θ is completely mixing if and only if it is exact (i.e., $\bigcap_{n=0}^{\infty} \theta^{-n}\Sigma = \{\emptyset, X\}$). See [17].

COROLLARY 5.2. *If P and Q are completely mixing Markov operators, then $P \times Q$ is completely mixing.*

COROLLARY 5.3. *If P is conservative and mixing, then $P \times Q$ is ergodic for every conservative and ergodic Q.*

PROOF. If P has no finite invariant measure, it is completely mixing [16], and Theorem 5.1 applies. If P has a finite invariant measure, it is equivalent to m. Since mixing implies mild mixing, $P \times Q$ is ergodic for every ergodic and conservative Q, by [8].

EXAMPLE 5.4. Products of conservative mixing Markov operators which are not mixing.

Take P mixing with invariant probability, but not completely mixing (e.g., P obtained by an invertible mixing transformation). Take Q completely mixing without a finite invariant measure. Then $P \times Q$ has no finite invariant measure, since Q has none. If $P \times Q$ were mixing, it would have been completely mixing, by [16], implying complete mixing of P which is false. (Note that $P \times Q$ is conservative and weak mixing, in this example, and satisfies also the conclusion of Corollary 5.3.)

REMARK. It is shown in [18] that for P conservative and mixing, $P \times P$ is mixing. This can also be proved using Theorem 5.1.

6. Mild mixing and Cartesian products

THEOREM 6.1. *Let θ be a mildly mixing transformation in (X, Σ, m). Then for every invertible ergodic and conservative σ (in $(Y, \mathcal{B}, \mu)$) $\theta \times \sigma$ is ergodic.*

PROOF. We assume that $L_1(X, \Sigma, m)$ is separable (see Lemma 4.5 for the reduction to this case). Let B be the unit ball in $L_\infty(X; \Sigma, m)$, which is compact metric in the weak-* topology. We may and do assume $\mu(Y) = 1$. Let $f(x, y)$ be invariant for $\theta \times \sigma$, and w.l.g. $\|f\|_\infty \leqq 1$. Define a map $F(y)$ from Y into B by $F(y)(x) = f(x, y)$. It is easy to check that F is measurable. Let $\{U_j\}$ be a covering of B by balls (in its w^* metric) of diameter $< 1/r$. Then $Y = \bigcup_j F^{-1}(U_j)$. For a.e. $y \in F^{-1}(U_j)$ there is an $n_r(y)$ such that $\sigma^{-n_r(y)}y \in F^{-1}(U_j)$, since σ^{-1} is conservative. Hence, for a.e. $y \in Y$, $F(\sigma^{-n_r(y)}y) \to F(y)$ weak-* (in B). Hence for every $u(x) \in L_1(X)$,

$$\int f(x, y)u(x)dm = \int F(y)(x)u(x)dm = \lim_{r\to\infty} \int F(\sigma^{-n_r(y)}y)(x)u(x)dm$$

$$= \lim_{r\to\infty} \int f(x, \sigma^{-n_r(y)}y)u(x)dm = \lim_{r\to\infty} \int f(\theta^{n_r(y)}x, y)u(x)dm,$$

for those $y \in Y$ such that $F(\sigma^{-n_r(y)}y) \to F(y)$ weak-* and $f(\theta^n x, y) = f(x, \sigma^{-n}y)$ for all n and a.e. x. Thus, for a.e. y fixed, $f(\theta^{n_r(y)}x, y) \to f(x, y)$ weak-*. Since θ is mildly mixing, $f(x, y)$ does not depend on x, or $f(x, y) = f_1(y)$. Now $f_1(\sigma y) = f_1(y)$, and ergodicity of σ implies that f is constant. Hence $\theta \times \sigma$ is ergodic.

COROLLARY 6.2. *An invertible transformation θ which has no finite invariant measure is not mildly mixing.*

PROOF. Let θ be an ergodic invertible transformation. We assume that θ is not the shift on the integers (which has unimodular eigenvalues and is not mildly mixing) and therefore θ is conservative. By the remark following Theorem 3.2, since θ has no finite invariant measure, there is an invertible conservative and ergodic σ (ergodicity of σ follows in the construction from Lemma 3.1) such that $\theta \times \sigma$ is not conservative. Hence $\theta \times \sigma$ cannot be ergodic (since it is not the shift on the integers). Theorem 6.1 shows θ cannot be mildly mixing.

COROLLARY 6.3. *Let θ be mildly mixing. Then for every ergodic conservative σ with σ-finite invariant measure, $\theta \times \sigma$ is ergodic.*

PROOF. Let σ_0 be the two-sided shift of σ. Then $\theta \times \sigma_0$ is ergodic. Hence so is $\theta \times \sigma$ (see Lemma 2.10).

COROLLARY 6.4. *Let P be mildly mixing. Then for every ergodic conservative Q with σ-finite invariant measure, $P \times Q$ is ergodic.*

REMARK. The result of [8] shows that if P has a finite invariant measure, Corollary 6.4 is true even if Q has no σ-finite invariant measure.

THEOREM 6.5. *If P is completely mixing and Q is mildly mixing, then $P \times Q$ is mildly mixing.*

PROOF. Let P be defined on $L_\infty(X, m)$, Q on $L_\infty(Y, \mu)$. Let $f \in L_\infty(X \times Y)$ satisfy $(P \times Q)^{n_i} f \to f$ weak-$*$ in $L_\infty(X \times Y)$.

Take $u \in L_1(X)$ with $\int u\,dm = 0$, and $v \in L_1(Y)$. Then

$$\begin{aligned}
&\left| \int\int u(x)v(y)(P \times Q)^n f\,dm\,d\mu \right| \\
&= \left| \int\int uP^n(x)vQ^n(y)f(x,y)\,dm(x)\,d\mu(y) \right| \\
&\leqq \int |vQ^n(y)| \left| \int uP^n(x)f(x,y)\,dm(x) \right| d\mu(y) \\
&\leqq \int |vQ^n(y)|\,\|uP^n\|_1 \|f\|_\infty\,d\mu \\
&\leqq \|v\|_1 \|f\|_\infty \|uP^n\|_1 \\
&\xrightarrow[n\to\infty]{} 0.
\end{aligned}$$

Hence $\int\int u(x)v(y)f(x,y)\,dm\,d\mu = 0 = \int u(x)[\int f(x,y)v(y)\,d\mu]\,dm$. Fix $v \in L_1(Y)$, and denote $h(x) = \int f(x,y)v(y)\,d\mu$. Since $\int u(x)h(x)\,dm = 0$ for every $u \in L_1(X)$ with $\int u\,dm = 0$, we have that $h(x)$ is constant a.e. on X. Denote the constant by $\alpha(v)$. Then $|\alpha(v)| \leqq \|f\|_\infty \|v\|_1$. Since $\alpha(v)$ is linear in v, there is a

$g \in L_\infty(Y)$ such that $\alpha(v) = \int v(y)g(y)d\mu$. Hence $f(x, y) = g(y)$ a.e. on $X \times Y$. The assumption yields that $Q^{n_i}g \to g$ weak-* in $L_\infty(Y)$, and mild mixing of Q implies that g is constant a.e., hence so is f.

EXAMPLE 6.6. A conservative mildly mixing transformation with infinite invariant measure and non-atomic tail field.

Let θ be exact conservative with σ-finite infinite invariant measure, and let σ be invertible probability-preserving and mild mixing (on a non-atomic space). Then $\theta \times \sigma$ has the required properties.

THEOREM 6.7. *Let θ be the two-sided shift of a conservative mildly mixing Markov operator P with σ-finite invariant measure. If σ is a conservative and ergodic transformation with σ-finite invariant measure such that $\theta \times \sigma$ is conservative, then $\theta \times \sigma$ is ergodic.*

PROOF. Let ρ be the two-sided shift (natural extension) of σ. Let θ_1 be the (one-sided) shift of P. Then $\theta \times \sigma$ conservative implies $\theta_1 \times \sigma$ conservative (Lemma 2.10), and by Corollary 6.4 (and Theorem 2.9) $P \times \sigma$ and $\theta_1 \times \sigma$ are conservative and ergodic. By Theorem 2.11 $\theta \times \rho$ is (conservative and) ergodic, so by Lemma 2.10 $\theta \times \sigma$ is ergodic.

REMARKS. (1) If θ has a finite invariant measure, the conditions of the theorem are equivalent to mild mixing (since the construction in [8] yields a transformation with σ-finite invariant measure). If the invariant measure is infinite, θ is *not* mildly mixing (by Corollary 6.2).

(2) In contrast to the finite invariant measure case, the condition on θ in Theorem 6.7 does not imply that $\theta \times \theta$ is ergodic, since it may fail to be conservative. Such an example is given by σ in Example 4.3 (where we take two-sided shifts of aperiodic Markov chains). However, Corollary 4.9 shows that θ must be weakly mixing.

EXAMPLE 6.8. A weakly mixing invertible transformation with an infinite σ-finite invariant measure, which does not satisfy the conclusion of Theorem 6.7.

Let ρ be an invertible weakly mixing transformation with an invariant probability, which is *not* mildly mixing (it is indicated in [8] how to construct such transformations). By [8] there is an invertible ergodic (and conservative) σ, preserving an infinite σ-finite measure, such that $\rho \times \sigma$ is not ergodic. But $\rho \times \sigma$ is conservative by Theorem 3.2. Let τ be the transformation constructed in Theorem 3.3, and let τ_1 be the two-sided shift of the chain in that proof (τ is its one-sided shift). Then τ_1 is conservative and weakly mixing (see Corollary 4.9) with σ-finite infinite invariant measure. Now $\sigma \times \tau$ is conservative by Theorem 3.3, and ergodic by Theorem 5.1. Hence $\sigma \times \tau_1$ is conservative and ergodic. We define $\theta = \tau_1 \times \rho$, which is weakly mixing by Corollary 4.8. Then $\theta \times \sigma = \tau_1 \times \rho \times \sigma$, which is not ergodic since $\rho \times \sigma$ is not ergodic. But $\theta \times \sigma \cong \rho \times (\sigma \times \tau_1)$, which is conservative by Theorem 3.2, since $\sigma \times \tau_1$ is conservative.

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